Some Elementary Mathematics of One Dimensional Integral Value Transformations and Expectations

Sk. Sarif Hassan

Email: sarimif@gmail.com

Institute of Mathematics & Applications

Bhubaneswar, India

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History and Birth of Integral Value Transformations

During **2008-2009**, Prof. P. Pal Choudhury and Dr. S Sahoo et al. were working on Carry Value Transformation (CVT) and Its Applications in Computer Sciences.

Understanding CVT and Birth of IVTs

Let us introduce an example before definition

Suppose, we want the CVT of the numbers $(13)_{10} \equiv (1101)_2$ and $(14)_{10} \equiv (1110)_2$. Both are 4-bit numbers. The carry value is computed as follows:

$$
\begin{array}{lr}
\underline{\text{Carry:} \quad 1\,1\,0\,0\,0} & \\
\text{Augend:} & 1\,1\,0\,1 \\
\underline{\text{Addend:}} & \underline{1\,1\,1\,0} \\
\text{XOR:} & 0\,0\,1\,1
\end{array}
$$

A Matrix with Self Similarity

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	2	0	2	0	2	0	2	0	2	0	2	0	2	0	2	0	2	0	2	0	2	0	2	0	2	0	2	0	2	0	2
2	0	0	4	4	0	0	4	4	0	0	4	4	0	0	4	4	0	0	4	4	0	0	4	4	0	0	4	4	0	0	4	4
3	0	2	4	6	0	2	4	6	0	2	4	6	0	2	4	6	0	2	4	6	0	2	4	6	0	2	4	6	0	2	4	6
4	0	0	0	0	8	8	8	8	0	0	0	0	8	8	8	8	0	0	0	0	8	8	8	8	0	0	0	0	8	8	8	8
5	0	2	0	2	8	10	8	10	0	2	0	2	8	10	8	10	0	2	0	2	8	10	8	10	0	2	0	2	8	10	8	10
6	0	0	4	4	8	8	12	12	0	0	4	4	8	8	12	12	0	0	4	4	8	8	12	12	0	0	4	4	8	8	12	12
7	0	2	4	6	8	10	12	14	0	2	4	6	8	10	12	14	0	2	4	6	8	10	12	14	0	2	4	6	8	10	12	14
8	0	0	0	0	0	0	0	0	16	16	16	16	16	16	16	16	0	0	0	0	0	0	0	0	16	16	16	16	16	16	16	16
9	0	2	0	2	0	2	0	2	16	18	16	18	16	18	16	18	0	2	0	2	0	2	0	2	16	18	16	18	16	18	16	18
10	0	0	4	4	0	0	4	4	16	16	20	20	16	16	20	20	0	0	4	4	0	0	4	4	16	16	20	20	16	16	20	20
11	0	2	4	6	0	2	4	6	16	18	20	26	16	18	20	26	0	2	4	6	0	2	4	6	16	18	20	26	16	18	20	26
12	0	0	0	0	8	8	8	8	16	16	16	16	24	24	24	24	0	0	0	0	8	8	8	8	16	16	16	16	24	24	24	24
13	0	2	0	2	8	10	8	10	16	18	16	18	24	26	24	26	0	2	0	2	8	10	8	10	16	18	16	18	24	26	24	26
14	0	0	4	4	8	8	12	12	16	16	20	20	24	24	28	28	0	0	4	4	8	8	12	12	16	16	20	20	24	24	28	28
15	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32
17	0	2	0	2	0	2	0	2	0	2	0	2	0	2	0	2	32	34	32	34	32	34	32	34	32	34	32	34	32	34	32	34
18	0	0	4	4	0	0	4	4	0	0	4	4	0	0	4	4	32	32	36	36	32	32	36	36	32	32	36	36	32	32	36	36
19	0	2	4	6	0	2	4	6	0	2	4	6	0	2	4	6	32	34	36	38	32	34	36	38	32	34	36	38	32	34	36	38
20	0	0	0	0	8	8	8	8	0	0	0	0	8	8	8	8	32	32	32	32	40	40	40	40	32	32	32	32	40	40	40	40
21	0	2	0	2	8	10	8	10	0	2	0	2	8	10	8	10	32	34	32	34	40	42	40	42	32	34	32	34	40	42	40	42
22	0	0	4	4	8	8	12	12	0	0	4	4	8	8	12	12	32	32	36	36	40	40	44	44	32	32	36	36	40	40	44	44
23	0	2	4	6	8	10	12	14	0	2	4	6	8	10	12	14	32	34	36	38	40	42	44	46	32	34	36	38	40	42	44	46
24	0	0	0	0	0	0	0	0	16	16	16	16	16	16	16	16	32	32	32	32	32	32	32	32	48	48	48	48	48	48	48	48
25	0	2	0	2	0	2	0	2	16	18	16	18	16	18	16	18	32	34	32	34	32	34	32	34	48	50	48	50	48	50	48	50
26	0	0	4	4	0	0	4	4	16	16	20	20	16	16	20	20	32	32	36	36	32	32	36	36	48	48	52	52	48	48	52	52
27	0	2	4	6	0	2	4	6	16	18	20	26	16	18	20	26	32	34	36	38	32	34	36	38	48	50	52	54	48	50	52	54
28	0	0	0	0	8	8	8	8	16	16	16	16	24	24	24	24	32	32	32	32	40	40	40	40	48	48	48	48	56	56	56	56
29	0	2	0	2	8	10	8	10	16	18	16	18	24	26	24	26	32	34	32	34	40	42	40	42	48	50	48	50	56	58	56	58
30	0	0	4	4	8	8	12	12	16	16	20	20	24	24	28	28	32	32	36	36	40	40	44	44	48	48	52	52	56	56	60	60
31	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50	52	54	56	58	60	62

Fig. 3: A Fractal Structure on using CVT of Different Integer Values in Binay Number System.

CVT in Ternary System (3-adic)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	3	0	0	3	0	0	3	0	0	3	0	0	3	0	0	3	0	0	3	0	0	3	0	0	3
2	0	3	3	0	3	3	0	3	3	0	3	3	0	3	3	0	3	3	0	3	3	0	3	3	0	3	3
3	0	0	0	0	0	0	9	9	9	0	0	0	0	0	0	9	9	9	0	0	0	0	0	0	9	9	9
4	0	0	3	0	0	3	9	9	12	0	0	3	0	0	3	9	9	12	0	0	3	0	0	3	9	9	12
5	0	3	3	0	3	3	9	12	12	0	3	3	0	3	3	9	12	12	0	3	3	0	3	3	9	12	12
6	0	0	0	9	9	9	9	9	9	0	0	0	9	9	9	9	9	9	0	0	0	9	9	9	9	9	9
7	0	0	3	9	9	12	9	9	12	0	0	3	9	9	12	9	9	12	0	0	3	9	9	12	9	9	12
8	0	3	3	9	12	12	9	12	12	0	3	3	9	12	12	9	12	12	0	3	3	9	12	12	9	12	12
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	27	27	27	27	27	27	27	27	27
0	0	0	3	0	0	3	0	0	3	0	0	3	0	0	3	0	0	3	27	27	30	27	27	30	27	27	30
1	0	3	3	0	3	3	0	3	3	0	3	3	0	3	3	0	3	3	27	30	30	27	30	30	27	30	30
2	0	0	0	0	0	0	9	9	9	0	0	0	0	0	0	9	9	9	27	27	27	27	27	27	36	36	36
3	0	0	3	0	0	3	9	9	12	0	0	3	0	0	3	9	9	12	27	27	30	27	27	30	36	36	39
4	0	3	3	0	3	3	9	12	12	0	3	3	0	3	3	9	12	12	27	30	30	27	30	30	36	39	39
5	0	0	0	9	9	9	9	9	9	0	0	0	9	9	9	9	9	9	27	27	27	36	36	36	36	36	36
6	0	0	3	9	9	12	9	9	12	0	0	3	9	9	12	9	12	12	27	27	30	36	36	39	36	39	39
7	0	3	3	9	12	12	9	12	12	0	3	3	9	12	12	9	12	12	27	30	30	36	39	36	36	39	39
8	0	0	0	0	0	0	0	0	0	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27
9	0	0	3	0	0		0	0	3	27	27	30	27	27	30	27	27	30	27	27	30	27	27	30	27	27	30
0	0	3	3	0	3	3	0	3	3	27	30	30	27	30	30	27	30	30	27	30	30	27	30	30	27	30	30
1	0	0	0	0	0	0	9	9	9	27	27	27	27	27	27	36	36	36	27	27	27	27	27	27	36	36	36
2	0	0	3	0	0	3	9	9	12	27	27	30	27	27	30	36	36	39	27	27	30	27	27	30	36	36	39
3	0	3	3	0	3	3	9	12	12	27	30	30	27	30	30	36	39	36	27	30	30	27	30	30	36	39	36
4	0	0	0	9	9	9	9	9	9	27	27	27	36	33	36	36	36	36	27	27	27	36	33	36	36	36	36
5	0	0	3	9	9	12	9	9	12	27	27	30	36	36	39	36	39	39	27	27	30	36	36	39	36	39	39
6	0	3	3	9	12	12	9	12	12	27	30	30	36	39	39	36	39	39	27	30	30	36	39	39	36	39	39

Fig. 4: A Fractal Structure on using CVT of Different Integer Values in Ternary Number System.

CVT Matrix (4 adic System)

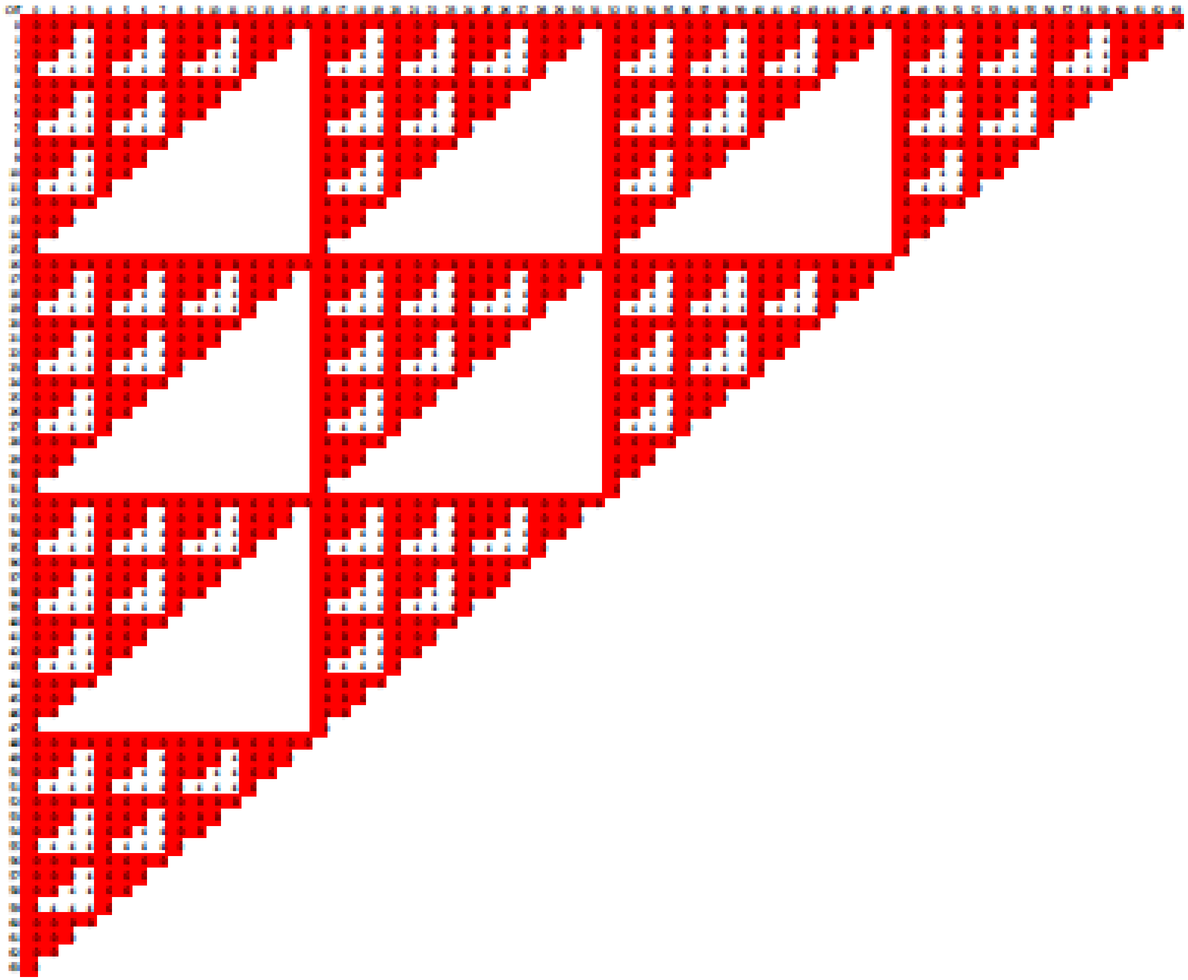

Fig. 5: A Fractal Structure on using CVT of Different Integer Values in 4-nary Number System.

As P diverges, Fractal Dimension Converges to Topological Dimension 2

Theorem IV.1. *The fractal dimension S_D converges to the topological dimension (Euclidian dimension) 2 as the base 'n' of the number system diverges to infinity.*

Proof. Let us try to find the limit when n tends to infinity

$$\lim_n S_D(n) = \lim_n \left\{ \log(\frac{n(n+1)}{2}) / \log n \right\} (\frac{\infty}{\infty} form)$$
$$= \lim \left\{ [\frac{2(2n+1)}{2n(n+1)}] / \frac{1}{n} \right\}, \text{by L'hospital rule}$$
$$= \lim \left\{ \frac{2n^2+n}{n^2+n} \right\} = 2.$$

So, starting from the binary number system the fractal dimension of the generated fractal will go on increasing with the increase of the base of the number system and finally it converges to the topological dimension 2. □

- Can we have some transformation (like CVT) which would lead us the following result
- As P diverges, Fractal Dimension Converges to Topological Dimension 1.

In Reply, YES, The functions is Known as EVTs (Extreme Value Transformations)

Matrix with Self Similarity

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1	3	3	5	5	7	7	9	9	11	11	13	13	15	15
2	2	3	2	3	6	7	6	7	10	11	10	11	14	15	14	15
3	3	3	3	3	7	7	7	7	11	11	11	11	15	15	15	15
4	4	5	6	7	4	5	6	7	12	13	14	15	12	13	14	15
5	5	5	7	7	5	5	7	7	13	13	15	15	13	13	15	15
6	6	7	6	7	6	7	6	7	14	15	14	15	14	15	14	15
7	7	7	7	7	7	7	7	7	15	15	15	15	15	15	15	15
8	8	9	10	11	12	13	14	15	8	9	10	11	12	13	14	15
9	9	9	11	11	13	13	15	15	9	9	11	11	13	13	15	15
10	10	11	10	11	14	15	14	15	10	11	10	11	14	15	14	15
11	11	11	11	11	15	15	15	15	11	11	11	11	15	15	15	15
12	12	13	14	15	12	13	14	15	12	13	14	15	12	13	14	15
13	13	13	15	15	13	13	15	15	13	13	15	15	13	13	15	15
14	14	15	14	15	14	15	14	15	14	15	14	15	14	15	14	15
15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15

Fig. 11: A Fractal Structure on using EVT of Different Integer Values in Binary Number System.

Matrix through EVT on 3-Adic System

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	1	1	2	4	4	5	7	7	8	10	10	11	13	13	14	16	16	17
2	2	2	2	5	5	5	8	8	8	11	11	11	14	14	14	17	17	17
3	3	4	5	3	4	5	6	7	8	12	13	14	12	13	14	15	16	17
4	4	4	5	4	4	5	7	7	8	13	13	14	13	13	14	16	16	17
5	5	5	5	5	5	5	8	8	8	14	14	14	14	14	14	17	17	17
6	6	7	8	6	7	8	6	7	8	15	16	17	15	16	17	15	16	17
7	7	7	8	7	7	8	7	7	8	16	16	17	16	16	17	16	16	17
8	8	8	8	8	8	8	8	8	8	17	17	17	17	17	17	17	17	17
9	9	10	11	12	13	14	15	16	17	9	10	11	12	13	14	15	16	17
10	10	10	11	13	13	14	16	16	17	10	10	11	13	13	14	16	16	17
11	11	11	11	14	14	14	17	17	17	11	11	11	14	14	14	17	17	17
12	12	13	14	12	13	14	15	16	17	12	13	14	12	13	14	15	16	17
13	13	13	14	13	13	14	16	16	17	13	13	14	13	13	14	16	16	17
14	14	14	14	14	14	14	17	17	17	14	14	14	14	14	14	17	17	17
15	15	16	17	15	16	17	15	16	17	15	16	17	15	16	17	15	16	17
16	16	16	17	16	16	17	16	16	17	16	16	17	16	16	17	16	16	17
17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17

Fig. 12: A Fractal Structure on using EVT of Different Integer Values in Ternay Number System.

As P diverges, Fractal Dimension Converges to Topological Dimension 1.

Theorem VIII.1. *The fractal dimension S_D converges to the topological dimension (Euclidian dimension) 1 as the base 'n' of the number system, diverges to infinity.*

Proof. Let us try to find the limit when n trends to infinity.

$$\lim_n S_D(n) = \lim_n \left\{ \frac{\log(2n-1)}{\log n} \right\} \left(\frac{\infty}{\infty} form\right)$$

$$= \lim \left\{ \left[\frac{2}{(2n-1)}\right] / \frac{1}{n} \right\} \text{ by L'Hospital rule}$$

$$= \lim \left\{ \frac{2n}{2n-1} \right\} = 1, \text{i.e.,} \lim_{n\to\infty} S_D(n) = 1.$$

So Far we Have used Two Variable P-Adic Functions in Generating Self-Similar Fractals

Immediate Questions

- Can we Extend the dimension of the underlying space where CVT, EVTs are acting?
- Answer is **YES**

This lead to the Birth of **Integral Value Transformations** in 2010.

Definition of IVT

$IVT^{p,k}{}_j : \mathbb{N}_0{}^K \rightarrow \mathbb{N}_0$

$IVT^{p,k}{}_j((n_1, n_2, \dots . n_k)=$

$(f_j(a_0{}^{n_1}, a_0{}^{n_2}, \dots, a_0{}^{n_k})\, f_j(a_1{}^{n_1}, a_1{}^{n_2}, \dots, a_1{}^{n_k}) \dots \dots f_j(a_{l-1}{}^{n_1}, a_{l-1}{}^{n_2}, \dots, a_{l-1}{}^{n_k}))_p = m$

where $n_1 = (a_0{}^{n_1} a_1{}^{n_1} \dots . a_{l-1}{}^{n_1})_p$, $n_2 = (a_0{}^{n_2} a_1{}^{n_2} \dots . a_{l-1}{}^{n_2})_p, \dots \dots$ $n_k = (a_0{}^{n_k} a_1{}^{n_k} \dots . a_{l-1}{}^{n_k})_p$

f_j: $\{0, 1, 2, \dots, p-1\}^k \rightarrow \{0,1,2, \dots, p-1\}$.

m is the decimal conversion from the p adic number.

An Example

For p=3, k=1. There are 3^{3^1} number of 1-dimension 3-variable functions two of which are given in the table below :

Variables	f_7	f_{16}
0	0	1
1	2	2
2	1	1

$x = 55 = (2001)_3$

$IVT^{3,1}{}_7(x) = (f_7(2)\, f_7(0) f_7(0) f_7(1))_3 = (1002)_3 = 29$

$IVT^{3,1}{}_{16}(x) = (f_{16}(2)\, f_{16}(0) f_{16}(0) f_{16}(1))_3 = (1112)_3 = 41$

Set of One Dimensional IVTs

Let us fix the domain of IVTs as $\mathbb{N}_0$ (k=1) and thus the above definition boils down to the following:

$$\mathrm{IVT}^{p,1}{}_j\,(x) = \big(f_j(x_n)\, f_j(x_{n-1}) \ldots \ldots \ldots f_j(x_1)\big)_p = m$$

where m is the decimal conversion from the p adic number, and $x = (x_n\, x_{n-1} \ldots \ldots x_1)_p$.

Let us denote the set of $\mathrm{IVT}^{p,1}{}_j$ as

$$T^{p,1} = \left\{ \mathrm{IVT}^{p,1}{}_j \;: \mathbb{N} \to \mathbb{N} \;\middle|\; \begin{array}{c} 0 \le j < p^p,\; \mathrm{IVT}^{p,1}{}_j\,(x) = \big(f_j(x_n)\, f_j(x_{n-1}) \ldots \ldots \ldots f_j(x_1)\big)_p = m \\ \text{where m is the decimal conversion from the p adic number} \\ \text{and } x = (x_n\, x_{n-1} \ldots \ldots x_1)_p \end{array} \right\}$$

Algebraic Structures on $T^{p,1}$

$(T^{p,1}, \oplus, \otimes)$ forms a **Commutative Ring** under the operations $\oplus$ and $\otimes$ defined as

$$(IVT^{p,1}{}_{j_1} \oplus IVT^{p,1}{}_{j_2})(x) = \left(f_{j_1}(x_n) \oplus_p f_{j_2}(x_n) \quad f_{j_1}(x_{n-1}) \oplus_p f_{j_2}(x_{n-1}) \ldots\ldots\ldots f_{j_1}(x_1) \oplus_p f_{j_2}(x_1)\right)_p \text{ and}$$

$$(IVT^{p,1}{}_{j_1} \otimes IVT^{p,1}{}_{j_2})(x) = \left(f_{j_1}(x_n) \otimes_p f_{j_2}(x_n) \quad f_{j_1}(x_{n-1}) \otimes_p f_{j_2}(x_{n-1}) \ldots\ldots\ldots f_{j_1}(x_1) \otimes_p f_{j_2}(x_1)\right)_p$$

where $x = (x_n\ x_{n-1} \ldots\ldots x_1)_p$ and $\oplus_p$ denotes addition modulo p and $\otimes_p$ denotes multiplication modulo p.

But the **multiplicative inverse of non-zero IVTs does not exist.**
So We have failed to proceed to **Field Structure**.

An Attempt to Make Field Structure

We already know that $(T^{p,1}, \oplus)$ where $\oplus$ is defined as

$$(\,IVT^{p,1}{}_{j_1} \oplus IVT^{p,1}{}_{j_2})(x) = \left(f_{j_1}(x_n) \oplus_p f_{j_2}(x_n)\;\; f_{j_1}(x_{n-1}) \oplus_p f_{j_2}(x_{n-1}) \ldots \ldots \ldots f_{j_1}(x_1) \oplus_p f_{j_2}(x_1)\right)_p$$

where $x = (x_n\; x_{n-1} \ldots \ldots x_1\,)_p$ is an abelian group. Our task is now to define an operation $\otimes$ in such a way that $(T^{p,1}, \otimes)$ forms an abelian group and follows the distributive laws.

Let us define an operation $\otimes$ as below:

$\otimes$: $T^{p,1} \times T^{p,1} \rightarrow T^{p,1}$ defined as

$$(\,IVT^{p,1}{}_{j_1} \otimes IVT^{p,1}{}_{j_2})(x) = \left(f_{j_1}(x_n) \otimes_p f_{j_2}(x_n)\;\; f_{j_1}(x_{n-1}) \otimes_p f_{j_2}(x_{n-1}) \ldots \ldots \ldots f_{j_1}(x_1) \otimes_p f_{j_2}(x_1)\right)_p$$

if either $IVT^{p,1}{}_{j_1} = \text{Identity}$ or $IVT^{p,1}{}_{j_2} = \text{Identity}$ or both

else $(\,IVT^{p,1}{}_{j_1} \otimes IVT^{p,1}{}_{j_2})(x) =$

$$\left(\delta\left(f_{j_1}(x_n) \otimes_p f_{j_2}(x_n)\right)\; \delta(f_{j_1}(x_{n-1}) \otimes_p f_{j_2}(x_{n-1})) \ldots \ldots \ldots \delta(f_{j_1}(x_1) \otimes_p f_{j_2}(x_1))\right)_p$$

where $\delta(x_i) = \begin{cases} x_i, & \text{if } x_i \neq 0 \\ 1, & \text{if } x_i = 0 \end{cases}$ where $x_i \in \mathbb{F}_p$

$(T^{p,1}, \otimes)$ is an abelian group but it does not follow the distributive properties and hence fails to be a field. Thus, unfortunately our efforts in this direction have been unsuccessful so far.

Immediate Expectations

- Would it be at all possible to have Field Structure on the set of IVTs?
- **Main Problem:** Distributive Law (Compatibility of Binary Operations).

So, Is there any protocol to design to make two binary operations Compatible (Distributive) on a set of interest?

- **Basic Question:** How can we ensure the **Existence or Non-Existence** about Group, Ring, Field (Algebraic Structures) on a set of interest?

Vector Space and Module Structure

$(T^{p,1}{}_{\#}, \oplus, \wedge)$ forms a **Vector space over a field** $\mathbb{F}_p$(**p is prime**) where ^ denotes scalar multiplication defined as $(c \wedge IVT^{p,1}{}_j)(x) = \left(c \otimes_p f_j(x_n) \quad c \otimes_p f_j(x_{n-1}) \ldots\ldots\ldots \quad c \otimes_p f_j(x_1)\right)_p$

where $x = (x_n\ x_{n-1} \ldots\ldots x_1)_p$ and $\otimes_p$ denotes multiplication modulo p.

Remark: When p is prime, $T^{p,1}$ is a finite vector space (with p^p functions) with $\dim(T^{p,1}) = p$ over the finite field $\mathbb{F}_p$. When p is composite number, $(T^{p,1}, \oplus, \wedge)$ forms a module over $\mathbb{F}_p$ which is a commutative ring with unity under addition and multiplication modulo p. Moreover, it is a *free module* since it has a basis which will be shown in the next slide.

Basis Functions for The Vector Space

The basis of $(T^{p,1}, \oplus, \wedge)$ is $\{ IVT^{p,1}_j$ such that $j = p^i$ where $i = 0,1,2,\dots p-1 \}$.

Thus for any $IVT^{p,1}_j \in T^{p,1}$, $IVT^{p,1}_j = a_0 \wedge IVT^{p,1}_{p^0} + a_1 \wedge IVT^{p,1}_{p^1} + a_2 \wedge IVT^{p,1}_{p^2} \dots\dots\dots\dots + a_{p-1} \wedge IVT^{p,1}_{p^{p-1}}$

where $a_i \in \mathbb{F}_p$ and $j = \sum_{i=0}^{p-1} a_i p^i$.

Illustration:

Basis functions of $T^{3,1}$ are $IVT^{3,1}_1$, $IVT^{3,1}_3$, $IVT^{3,1}_9$

In $T^{3,1}$, we can write $IVT^{3,1}_{21} = a_0 \wedge IVT^{3,1}_{3^0} + a_1 \wedge IVT^{3,1}_{3^1} + a_2 \wedge IVT^{3,1}_{3^2}$ where $a_i \in \mathbb{F}_3, i = 0,1,2$

and $21 = a_0 + 3a_1 + 9a_2$. So we must have $a_0 = 0$, $a_1 = 1$, $a_2 = 2$.

Thus we must have

$IVT^{3,1}_{21} = 0 \wedge IVT^{3,1}_{3^0} + 1 \wedge IVT^{3,1}_3 + 2 \wedge IVT^{3,1}_9$

What is now?

This Mathematics of IVTs is one-fold research work of Home of Mathematical Genomics (HMG-ISI Kolkata) (www.isical.ac.in/~hmg).

There is another one which is Genomics through Mathematics.

Thanks